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Nonlinear equations invariant under Poincaré, similitude and conformal group in three-dimensional spacetime

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Abstract. This paper is devoted to a systematic construction of second-order differential equations invariant under the Poincaré, similitude and conformal groups in three-dimensional spacetime. A classification of all possible realizations of the Lie algebras under the action of the group of local diffeomorphisms of $\mathbb{R}^4$ is presented. Then by means of the differential invariants the most general invariant differential equations of second order are constructed.

1. Introduction

In [1], the problem of constructing second-order differential equations invariant under the Poincaré, similitude and conformal groups in two-dimensional Minkowski space was considered. In this paper we discuss the same problem in three-dimensional spacetime leading to different types of invariant equations from those arising in the two-dimensional case.

The motivation for constructing equations invariant under a prescribed symmetry group, also known as inverse problems, is two-fold. First, we classify equations admitting some physically important symmetry groups. Secondly, knowing that by construction these equations are invariant we are able to solve them or at least find certain explicit particular solutions by using the symmetry reduction method. Consequently, the inverse problem appears to be of interest both from a mathematical and physical point of view.

The paper is organized as follows. In section 2, we give an overview of the differential invariants and invariant equations. In section 3, we obtain the realizations of the Lie algebras $\mathrm{p}(2,1)$, $\mathrm{s}(2,1)$ and $\mathrm{c}(2,1)$ by vector fields in four variables (x, y, t, u). In section 4, we obtain the second-order differential invariants and hence the invariant equations.

Throughout the paper, the requirement of symmetry will be meant in the sense that if a second-order equation is invariant under some one-parameter group, then this equation is annihilated by the second-order prolongations everywhere, not only on the solution set of the equation.

2. Mathematical preliminaries and notation

A detailed and modern discussion of the Lie theory of symmetry groups of differential equations can be found in [2, 3]. Let G be a local Lie group of transformations acting

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on the space $X \otimes U$ of independent and dependent variables. Let $G^{(n)} = \mathrm{pr}^{(n)}G$ denote the prolonged group action on the jet space J^n whose coordinates are denoted by $(x, u^{(n)})$. The space of infinitesimal generators of G, i.e. its Lie algebra will be denoted by $\mathfrak{g}$ with associated prolongation $\mathfrak{g}^{(n)} = \mathrm{pr}^{(n)}\mathfrak{g}$.

We present a number of definitions and theorems which will be used throughout the paper.

Definition 1. A differential invariant of order $r \leqslant n$ is a scalar function $I : J^n \to \mathbb{R}$ which satisfies

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

for all $g^{(n)} \in G^{(n)}$ and all $(x, u^{(n)}) \in J^n$.

A relative differential invariant is a differential function which is invariant, up to a factor, under the prolonged group action. We denote the set of all differential invariants of order k by $\mathcal{I}_k(\mathfrak{g})$.

Definition 2. A set of kth order differential invariants $\{I_1, I_2, \dots, I_r\}$ is said to form a local basis of $\mathcal{I}_k(\mathfrak{g})$ if every differential invariant $I \in \mathcal{I}_k(\mathfrak{g})$ can be locally represented as $I = F(I_1, I_2, \dots, I_r)$ for some smooth function F and the functions $I_1, I_2, \dots, I_r$ are functionally independent.

Proposition 3. If $I_1, I_2, \dots I_r \in \mathcal{I}_k(\mathfrak{g})$ and $F(x_1, x_2, \dots, x_r)$ is an arbitrary smooth function of r variables then $F(I_1, I_2, \dots I_r) \in \mathcal{I}_k(\mathfrak{g})$.

Proposition 4. A function $I : J^n \to \mathbb{R}$ is a differential invariant for a connected transformation group G if and only if for some differential function μ

$$\mathrm{pr}^{(n)}\boldsymbol{v}(I) = \mu(x, u^{(n)})I$$

for every prolonged infinitesimal generator $\mathrm{pr}^{(n)}\boldsymbol{v} \in \mathfrak{g}^{(n)}$.

When $\mu = 0$, I is called an absolute or ordinary invariant. Differential invariants permit us to construct a number of classes of differential equations with a prescribed symmetry. Relative invariants are equally important in the construction of invariant equations. A classification of all wave equations using relative invariants has been carried out in [4]

In the following invariant will mean absolute invariant.

Theorem 5. Let G be a transformation group and let $\{I_1, I_2, \dots I_k\}$ be a complete system of functionally independent nth order differential invariants on an open subset $V^n \subset J^n$. A system of differential equations admit G as a symmetry group iff, when restricted to the subset V^n, it can be written in terms of the differential invariants:

$$\Delta_\nu(x, u^{(n)}) = F_\nu(I_1, I_2, \dots I_k) = 0 \qquad \nu = 1, 2, \dots, l.$$

Lemma 6. Let $\boldsymbol{v}$ be a vector field defined on M. If x_0 is not a singularity of $\boldsymbol{v}$, so $\boldsymbol{v}|_{x_0} \neq 0$, then there exists local rectifying coordinates $y = (y_1, y_2, \dots, y_n)$ near x_0 such that $\boldsymbol{v} = \partial_{y_1}$.

3. Realization of the Lie algebras by vector fields

3.1. Poincaré, similitude and conformal groups in M(2, 1)

The group of isometries of the $(2 + 1)$-dimensional Minkowski space $\mathrm{M}(2, 1)$ with coordinates (t, x, y) and metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$, i.e. the group of point transformations leaving the distance between two points in this space invariant (norm preserving transformations) is the Poincaré group $\mathrm{P}(2, 1)$, also called inhomogeneous

Lorentz group. The group extended by dilations is called similitude or extended Poincaré group and has a structure

$$\mathrm{S}(2,1) = D \rhd (\mathrm{SL}(2,\mathbb{R}) \rhd T_3)$$

where T_3 are three-dimensional translations and $\mathrm{SL}(2,\mathbb{R})$ is the special linear group isomorphic to $\mathrm{O}(2,1)$ and $\rhd$ denotes the semidirect product. The similitude group is a subgroup of the conformal group of spacetime, i.e. the group of all transformations leaving the Lorentz metric form invariant. The conformal group is also locally isomorphic to the de Sitter group $\mathrm{SO}(3,2)$.

The conformal group $\mathrm{C}(2,1)$ is generated by spacetime translations P_μ, $\mu = 0,1,2$, Lorentz boosts K_i, $i = 1,2$, infinitesimal rotation L_3, dilation D and proper conformal transformations C_μ, $\mu = 0,1,2$. These generators satisfy nonzero commutation relations

$$\begin{aligned}
&[K_i, P_0] = P_i && [P_\mu, D] = P_\mu\\
&[P_0, C_0] = 2D && [P_0, C_i] = 2K_i\\
&[P_1, K_1] = -P_0 && [P_1, L_3] = -P_2\\
&[P_1, C_0] = -2K_1 && [P_1, C_1] = -2D\\
&[P_1, C_2] = -2L_3 && [P_2, K_2] = -P_0\\
&[P_2, L_3] = P_1 && [P_2, C_0] = -2K_2\\
&[P_2, C_1] = 2L_3 && [P_2, C_2] = -2D\\
&[K_1, K_2] = -L_3 && [K_1, L_3] = -K_2\\
&[K_1, C_0] = C_1 && [K_1, C_1] = C_0\\
&[K_2, C_0] = C_2 && [K_2, L_3] = K_1\\
&[K_2, C_2] = C_0 && [L_3, C_1] = C_2\\
&[L_3, C_2] = -C_1 && [D, C_\mu] = C_\mu.
\end{aligned} \tag{3.1}$$

3.2. The Lie algebras realized by vector fields

We shall classify realizations of the Lie algebras $\mathrm{p}(2,1)$ and its subalgebras $\mathrm{s}(2,1)$ and $\mathrm{c}(2,1)$ associated with the conformal group and its subgroups in terms of vector fields

$$\boldsymbol{v} = \xi(x,y,t,u)\partial_x + \eta(x,y,t,u)\partial_y + \tau(x,y,t,u)\partial_t + \phi(x,y,t,u)\partial_u \tag{3.2}$$

up to diffeomorphisms

$$\begin{aligned}
&\tilde{x} = X(x,y,t,u) && \tilde{y} = Y(x,y,t,u)\\
&\tilde{t} = T(x,y,t,u) && \tilde{u} = U(x,y,t,u).
\end{aligned} \tag{3.3}$$

We first realize the Abelian algebra $\mathfrak{t}_3 = \{P_0, P_1, P_2\}$ in coordinates (t,x,y,u). According to the rectification lemma 6 we can transform P_0 which is of the form (3.2) to the standard form ∂_t. On using a transformation of the form (3.3) leaving ∂_t invariant, one can transform P_1 to ∂_x. Similarly, a transformation leaving $\{\partial_t, \partial_x\}$ invariant leads to the translation ∂_y. Finally, we have the generators of coordinate translations

$$\{P_0 = \partial_t, P_1 = \partial_x, P_2 = \partial_y\} \tag{3.4}$$

as the realization of the three-dimensional algebra $\mathfrak{t}_3$. From the commutation relations (3.1) involving P_μ and K_1, K_2 and L_3, the form of K_1, K_2 and L_3 is restricted to

$$K_1 = (-t + A_1(u))\partial_x + B_1(u)\partial_y + (-x + C_1(u))\partial_t + D_1(u)\partial_u \tag{3.5}$$

$$K_2 = A_2(u)\partial_x + (-t + B_2(u))\partial_y + (-y + C_2(u))\partial_t + D_2(u)\partial_u \tag{3.6}$$

$$L_3 = (y + A_3(u))\partial_x + (-x + B_3(u))\partial_y + C_3(u)\partial_t + D_3(u)\partial_u. \tag{3.7}$$

Furthermore, a transformation leaving P_μ, $\mu = 0, 1, 2$ invariant, namely

$$\tilde{x} = x + \alpha(u) \qquad \tilde{y} = y + \beta(u) \qquad \tilde{t} = t + \gamma(u) \qquad \tilde{u} = \delta(u)$$

with appropriate choices of α, β, γ and δ, reduces K_1 to precisely one of the following:

$$K_1 = -t\partial_x - x\partial_t + B_1(u)\partial_y \tag{3.8a}$$

$$K_1 = -t\partial_x - x\partial_t + u\partial_u. \tag{3.8b}$$

Further, performing a transformation leaving $\{P_0, P_1, P_2, K_1\}$ invariant

$$\tilde{x} = x \qquad \tilde{t} = t \qquad \tilde{y} = y + \rho(u) \qquad \tilde{u} = u$$

we can set $C_2 = 0$ or $B_2 = 0$ in K_2 of (3.6). If the commutation relations between K_1, K_2 and L_3, i.e.

$$[K_1, K_2] = -L_3 \qquad [K_1, L_3] = -K_2 \qquad [K_2, L_3] = K_1$$

are imposed we find the following realizations

$$P^{\mathrm{I}} : \{P_0 = \partial_t, P_1 = \partial_x, P_2 = \partial_y, K_1 = -t\partial_x - x\partial_t, K_2 = -t\partial_y + (-y + C_2)\partial_t, \\ L_3 = (y - C_2)\partial_x - x\partial_y\} \tag{3.9a}$$

$$P^{\mathrm{II}} : \Bigg\{P_0 = \partial_t, P_1 = \partial_x, P_2 = \partial_y, K_1 = -t\partial_x - x\partial_t + u\partial_u, \\ K_2 = -t\partial_y - y\partial_t + \left(a - \frac{u^2}{4a}\right)\partial_u, \\ L_3 = y\partial_x - x\partial_y + \left(a + \frac{u^2}{4a}\right)\partial_u, a \neq 0\Bigg\}. \tag{3.9b}$$

The first realization characterizes a class of algebras corresponding to an arbitrary function $C_2(u)$. If we restrict ourselves to the fibre preserving transformations, namely transformations in which the change in the independent variables are unaffected by the dependent variables, then a standard realization which we will denote by P is obtained.

We now extend the realization (3.9) to that of the similitude algebra $\mathrm{s}(2, 1)$. If we add a dilation generator to the standard realization P and again use the commutation relations involving D, we find

$$D = x\partial_x + y\partial_y + t\partial_t + a(u)\partial_u \qquad a(u) \text{ arbitrary.} \tag{3.10}$$

On invoking a transformation

$$\tilde{x} = x \qquad \tilde{t} = t \qquad \tilde{y} = y \qquad \tilde{u} = \sigma(u)$$

we can transform D to either of the following

$$D = x\partial_x + y\partial_y + t\partial_t \tag{3.11a}$$

$$D = x\partial_x + y\partial_y + t\partial_t + u\partial_u. \tag{3.11b}$$

Obviously, $\mathrm{p}(2, 1)$ generators remain invariant under this transformation. In conclusion, we obtain two inequivalent realizations of the similitude algebra represented by

$$S^{\mathrm{I}} : \{\{P\}, D = x\partial_x + y\partial_y + t\partial_t\} \tag{3.12a}$$

$$S^{\mathrm{II}} : \{\{P\}, D = x\partial_x + y\partial_y + t\partial_t + u\partial_u\}. \tag{3.12b}$$

Similarly, the realization (3.9b) leads to

$$S^{\mathrm{III}} : \{\{P^{\mathrm{II}}\}, D = x\partial_x + y\partial_y + t\partial_t\}. \tag{3.12c}$$

We should mention that if we leave P unchanged but transform the dilation D of (3.10) into

$$\tilde{D} = x\partial_x + y\partial_y + t\partial_t + \frac{2}{1-k}u\partial_u \qquad (k \neq 1) \tag{3.13}$$

and

$$\hat{D} = x\partial_x + y\partial_y + t\partial_t - \frac{2}{m}\partial_u \qquad (m \neq 0) \tag{3.14}$$

then the corresponding realizations $\tilde{S}$ and $\hat{S}$ represented by

$$\tilde{S} : \left\{ \{P\}, \tilde{D} = x\partial_x + y\partial_y + t\partial_t + \frac{2}{1-k}u\partial_u \right\} \tag{3.15a}$$

$$\hat{S} : \left\{ \{P\}, \hat{D} = x\partial_x + y\partial_y + t\partial_t - \frac{2}{m}\partial_u \right\} \tag{3.15b}$$

will permit us to construct certain invariant equations of physical importance.

Finally, the realizations of the algebra s(2, 1) can be extended to c(2, 1) by adding generators of the proper conformal transformations C_μ, $\mu = 0, 1, 2$. We only present the final results:

$$\begin{aligned} K^{\mathrm{I}} : \{\{S^{\mathrm{I}}\}, C_0 &= 2xt\partial_x + 2yt\partial_y + (x^2 + y^2 + t^2)\partial_t, \\ C_1 &= (t^2 + x^2 - y^2)\partial_x + 2xy\partial_y + 2xt\partial_t, \\ C_2 &= 2xy\partial_x + (t^2 - x^2 + y^2)\partial_y + 2yt\partial_t\} \end{aligned} \tag{3.16a}$$

$$\begin{aligned} K^{\mathrm{II}} : \{\{S^{\mathrm{II}}\}, C_0 &= 2xt\partial_x + 2yt\partial_y + (x^2 + y^2 + t^2)\partial_t + 2tu\partial_u, \\ C_1 &= (t^2 + x^2 - y^2)\partial_x + 2xy\partial_y + 2xt\partial_t + 2xu\partial_u, \\ C_2 &= 2xy\partial_x + (t^2 - x^2 + y^2)\partial_y + 2yt\partial_t + 2yu\partial_u\}. \end{aligned} \tag{3.16b}$$

$$\begin{aligned} K^{\mathrm{III}} : \Bigg\{\{S^{\mathrm{III}}\}, C_0 &= 2xt\partial_x + 2yt\partial_y + (x^2 + y^2 + t^2)\partial_t + 2\left[-ux + \left(\frac{u^2}{4a} - a\right)y\right]\partial_u, \\ C_1 &= (y^2 - x^2 - t^2)\partial_x - 2xy\partial_y - 2xt\partial_t + 2\left[tu + \left(\frac{u^2}{4a} + a\right)y\right]\partial_u, \\ C_2 &= -2xy\partial_x + (x^2 - y^2 - t^2)\partial_y - 2yt\partial_t \\ &\quad + 2\left[\left(-\frac{u^2}{4a} + a\right)t - \left(\frac{u^2}{4a} + a\right)x\right]\partial_u\Bigg\}. \end{aligned} \tag{3.16c}$$

An extension of realizations (3.15*a*)–(3.15*b*) to conformal transformations can be written as

$$\begin{aligned} \tilde{K} : \Bigg\{\{\tilde{S}\}, C_0 &= 2xt\partial_x + 2yt\partial_y + (x^2 + y^2 + t^2)\partial_t + \frac{4}{1-k}tu\partial_u, \\ C_1 &= (t^2 + x^2 - y^2)\partial_x + 2xy\partial_y + 2xt\partial_t + \frac{4}{1-k}ux\partial_u, \\ C_2 &= 2xy\partial_x + (t^2 - x^2 + y^2)\partial_y + 2yt\partial_t + \frac{4}{1-k}uy\partial_u, k \neq 1\Bigg\} \end{aligned} \tag{3.17a}$$

$$\begin{aligned} \hat{K} : \Bigg\{\{\hat{S}\}, C_0 &= 2xt\partial_x + 2yt\partial_y + (x^2 + y^2 + t^2)\partial_t - \frac{4}{m}t\partial_u, \\ C_1 &= (t^2 + x^2 - y^2)\partial_x + 2xy\partial_y + 2xt\partial_t - \frac{4}{m}x\partial_u, \\ C_2 &= 2xy\partial_x + (t^2 - x^2 + y^2)\partial_y + 2yt\partial_t - \frac{4}{m}y\partial_u, m \neq 0\Bigg\}. \end{aligned} \tag{3.17b}$$

All transformations corresponding to these generators are fibre preserving.

4. Differential invariants and invariant equations

Once the Lie algebra of the assumed symmetry group is realized in terms of vector fields on the space $X \otimes U$ of independent and dependent variables, we can proceed to the construction of invariant equations. To this end, we first assume *a priori* a general differential equation written in terms of the jet variables defined on the jet space J^n. Next we need to find the nth prolongations of the vector field. Since our primary interest is in the derivation of relativistically invariant scalar equations we will limit ourselves to the most general second-order ($n = 2$) partial differential equation (PDE) of the form

$$F(x, y, t, u, u_i, u_{ij}) = 0 \qquad i, j \in \{x, y, t\} \tag{4.1}$$

invariant under the assumed group. One might as well consider more general equations and construct higher-order invariant equations than in our case.

According to theorem 5, we can express equations invariant under a given group in terms of the invariants of the prolonged group action, which require the calculation of the second-order prolongations. Hence an invariant equation will have the form

$$F(I_1, I_2, \ldots, I_k) = 0 \tag{4.2}$$

where $\{I_1, I_2, \ldots, I_k\}$ forms a basis of $\mathcal{I}_2(\mathfrak{g})$.

The second-order prolongations of the translation algebra $\mathfrak{t}_3$ are immediate:

$$\mathrm{pr}^{(2)} P_\mu = P_\mu \qquad \mu = 0, 1, 2. \tag{4.3}$$

However, the other prolongations are more complicated. For example, the second prolongation of K_1 is

$$\mathrm{pr}^{(2)} K_1 = K_1 - u_t \partial_{u_x} - u_x \partial_{u_t} - 2u_{xt} \partial_{u_{xx}} - u_{yt} \partial_{u_{xy}} - (u_{xx} + u_{tt}) \partial_{u_{xt}} - u_{xy} \partial_{u_{yt}} - 2u_{xt} \partial_{u_{tt}}. \tag{4.4}$$

In order to find invariants one must solve a system of first-order differential equations

$$\mathrm{pr}^{(2)} \boldsymbol{v}(F) = 0 \tag{4.5}$$

for each vector field $\boldsymbol{v}$ chosen from the algebra and F having the form (4.1).

In the following sections, we compute the differential invariants and the invariant equations running through each group previously realized, respectively.

4.1. Poincaré invariant equations

Let us find the differential invariants of the standard realization of $\mathrm{p}(2, 1)$. From the requirement of invariance under the algebra of translations $\mathfrak{t}_3$ it follows that F should be independent of the variables x, y, t. When $\mathrm{pr}^{(2)} \boldsymbol{v}(F) = 0$ is solved, nine functionally independent invariants are found:

$$\begin{aligned}
&I_1 = u \qquad I_2 = u_y \qquad I_3 = u_{yy} \\
&I_4 = u_t^2 - u_x^2 \qquad I_5 = u_{yt}^2 - u_{xy}^2 \\
&I_6 = u_{tt} - u_{xx} \qquad I_7 = (u_t - u_x)^2 (u_{xx} + 2u_{xt} + u_{tt}) \\
&I_8 = (u_t + u_x)^2 (u_{xx} - 2u_{xt} + u_{tt}) \\
&I_9 = (u_{xy} + u_{yt})(u_t - u_x).
\end{aligned} \tag{4.6}$$

Since we wish to find the joint invariants of $p(2,1)$ we next express the vector field $\mathrm{pr}^{(2)}K_2$ in terms of the invariants of $\mathrm{pr}^{(2)}K_1$ as coordinates, namely $\{I_1,\ldots,I_9\}$. If we let $F(I_1,I_2,\ldots,I_9)$ act on $\mathrm{pr}^{(2)}K_2$ and re-express the resulting expression using the invariants I_μ, $\mu=1,\ldots 9$ we find only three invariants, namely

$$J_1=u \qquad J_2=I_4-I_2^2 \qquad J_3=I_6-I_3.$$

In terms of the original jet variables they can be written as

$$J_1=u \qquad J_2=(\nabla u)^2=u_t^2-u_x^2-u_y^2 \qquad J_3=\Box u=u_{tt}-u_{xx}-u_{yy}$$

where $\Box$ is the d'Alembert operator in $M(2,1)$ with the signature $(+,-,-,-)$. Since $\mathrm{pr}^{(2)}L_3=[\mathrm{pr}^{(2)}K_2,\mathrm{pr}^{(2)}K_1]$, every joint differential invariant of $\{K_1,K_2\}$ will simultaneously be an invariant of L_3. Hence we conclude that J_1,J_2,J_3 are invariants of the standard Poincaré realization and the most general Poincaré invariant equation has the form

$$F(u,\Box u,(\nabla u)^2)=0 \tag{4.7}$$

where F is an arbitrary function of its arguments. This includes the nonlinear Klein–Gordon (d'Alembert) equation

$$\Box u=H(u) \tag{4.8a}$$

and the first-order relativistic Hamilton–Jacobi equation

$$(\nabla u)^2+V(u)=E. \tag{4.8b}$$

In passing, let us mention that the d'Alembert-eikonal system

$$\Box u=0 \qquad (\nabla u)^2=0$$

for which a general solution exists arises naturally when solving the problem of reducing (4.8*a*) to ordinary differential equations (ODEs) [5].

In particular, setting $J_3/J_1=\lambda$ gives the linear equation

$$\Box u=\lambda u \qquad \lambda\in\mathbb{R}.$$

Furthermore, an eikonal (nonlinear) and linear wave equation are obtained by setting $J_2=0$ and $J_3=0$.

For the less standard realization P^{II}, there exists no equation invariant under the full group. Nevertheless, for certain subgroups a number of invariant equations can be obtained. Among them, the most remarkable ones are $\Box u=\lambda u$, $(\nabla u)^2=\mu u^2$ and $u_{yt}^2-u_{xy}^2=\nu u^2$.

4.2. Equations invariant under similitude group

The addition of the dilational invariance to Poincaré algebra reduces the Poincaré invariants by one. The first realization S^{I} of s(2,1) yields the following invariants

$$\tilde{J}_1=J_1=u \qquad \tilde{J}_2=\frac{\Box u}{(\nabla u)^2}$$

and the invariant equation becomes

$$F\left(u,\frac{\Box u}{(\nabla u)^2}\right)=0. \tag{4.9}$$

This equation has the form

$$\Box u=(\nabla u)^2h(u) \tag{4.10}$$

for some arbitrary function h. Similarly the second realization S^{II} gives rise to the invariant equation

$$F(u\Box u, (\nabla u)^2) = 0.$$

There exist two invariants of $\tilde{S}$:

$$\tilde{I}_1 = u^{-k}\Box u \qquad \tilde{I}_2 = u^{-(k+1)}(\nabla u)^2 \qquad k \neq 1.$$

The invariant equation is

$$F(\tilde{I}_1, \tilde{I}_2) = 0. \tag{4.11}$$

In particular, $\tilde{I}_1 = \lambda$ (constant) will give the nonlinear Klein–Gordon equation

$$\Box u = \lambda u^k. \tag{4.12}$$

For $\hat{S}$ invariants are:

$$\hat{I}_1 = \mathrm{e}^{-mu}\Box u \qquad \hat{I}_2 = \mathrm{e}^{-mu}(\nabla u)^2 \qquad m \neq 0$$

and the invariant equation is

$$F(\hat{I}_1, \hat{I}_2) = 0. \tag{4.13}$$

Setting $\hat{I}_1 = \lambda$ gives

$$\Box u = \lambda \mathrm{e}^{mu} \qquad m \neq 0 \tag{4.14}$$

which is the scalar curvature equation in $\mathrm{M}(2, 1)$.

4.3. Conformally invariant equations

Since $\mathrm{c}(2, 1)$ contains $\mathrm{s}(2, 1)$ the conformal invariants will be constructed using $\mathrm{s}(2, 1)$ invariants $\tilde{J}_1$ and $\tilde{J}_2$. Again, due to the relation $\mathrm{pr}^{(2)}C_i = [\mathrm{pr}^{(2)}K_i, \mathrm{pr}^{(2)}C_0]$, $(i = 1, 2)$ it is sufficient to solve only $\mathrm{pr}^{(2)}C_0(F) = 0$ and the result will be an invariant of the full conformal group.

On imposing the condition $\mathrm{pr}^{(2)}C_0(F) = 0$ will reduce them to a single one. The only equation invariant under K^{I} is $(\nabla u)^2 = 0$. The realization K^{II} has a single invariant $\Sigma = 2I - 3J$ and hence the invariant equation is

$$\Box u - \tfrac{3}{2}(\nabla u)^2 = \lambda \text{ (constant)}. \tag{4.15}$$

The condition $\mathrm{pr}^{(2)}C_0(F) = 0$ with C_0 as in $\tilde{K}$ can be written in terms of the $\tilde{S}$ invariants as

$$u_t u^{-k}\left(4\frac{\partial F}{\partial I} + (5-k)\frac{\partial F}{\partial J}\right) = 0.$$

Hence, there is only one invariant $\Sigma = (5-k)\tilde{I}_1 - 4\tilde{I}_2$ being automatically an invariant of C_1 and C_2, which leads to the invariant equation

$$u^{-k}\Box u - \frac{(5-k)}{4}u^{-(k+1)}(\nabla u)^2 = \lambda. \tag{4.16}$$

In particular, when $k = 5$ we obtain

$$\Box u = \lambda u^5 \qquad \lambda \in \mathbb{R} \tag{4.17}$$

containing the linear wave equation $\Box u = 0$. We remark that, more generally, the equation $\Box u = \lambda u^p$, $p = (n+3)/(n-1)$ is conformally invariant in $\mathrm{M}(n, 1)$ with $n > 1$ and a corresponding equation in $\mathrm{M}(1, 1)$ does not come up.

Likewise, the only invariant of $\hat{K}$ is $\mathrm{e}^{-mu}(\Box u + \frac{m}{4}(\nabla u)^2)$ and leads to the invariant equation

$$\Box u + \frac{m}{4}(\nabla u)^2 = \lambda \mathrm{e}^{mu} \qquad m \neq 0.$$

5. Conclusions

The results of this paper can be summed up as follows. We showed that there exist two inequivalent realizations of the Poincaré algebra $\mathrm{p}(2,1)$. The first one depending on arbitrary functions of dependent variable gives rise to a natural realization when restricted to fibre preserving transformations. The other is a less standard realization P^{II} of (3.9*b*). We find three inequivalent fibre preserving realizations of the similitude algebra given by S^{I}, S^{II} and S^{III}. We remark that two realizations $\tilde{S}$ and $\hat{S}$ which can be transformed by a point transformation to S^{II} provides physically important equations. The conformal algebra allows three inequivalent realizations. Again, we transformed the second realization K^{II} to $\tilde{K}$ and $\hat{K}$ for catching physically important invariant equations. In section 4, we obtained invariant second-order equations of the form (4.1). Since there is no restriction on the invariance condition (4.5), i.e. the equation we wish to construct is annihilated only on the solution set, here we do not obtain evolution-type equations. Instead, we found rather general relativistically invariant equations.

When compared with the results of [1], again we find several realizations some of which are physically less obvious and as in the two-dimensional case include more standard ones as special cases. As is to be expected, in both dimensions the corresponding invariant equations include linear and nonlinear Klein–Gordon- and eikonal-type equations. It is interesting to note that in contrast to the two-dimensional case the conformally invariant equations here are quite specific, namely we do not have a class of equations, but rather a single equation. In particular we obtained $\Box u = \lambda u^5$ as a conformal invariant equation which does not have a counterpart in $\mathrm{M}(1,1)$.

As a by-product of the classification of invariant equations one can perform symmetry reductions of the invariant equations obtained in section 4 and find group invariant solutions. For instance, using the results of classification of all subgroups of the similitude group in $\mathrm{M}(2,1)$ given in [6] we can classify reduced equations which result in PDEs in two variables and ODEs corresponding to one- and two-dimensional subgroups with generic orbits of codimension two and one respectively. Next, we can perform a singularity analysis for the second-order ODEs and pick out those of having Painlevé property and hence obtain exact solutions. We plan to return to this problem in the near future.

Finally, we finish by stating that it is always desirable to have a classification of equations admitting a given group at one's disposal. For, by construction, they are invariant under the group and symmetry methods and are immediately applicable to study these equations from several respects such as finding symmetry reductions, hence finding exact solutions, identifying integrable equations and establishing a connection between integrability and symmetry.

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